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Interior Hölder continuity for viscosity solutions of fully nonlinear second-order uniformly elliptic PDEs with measurable ingredients

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1 Introduction

In this note, we obtain the Harnack inequality for "weak" solutions of the following fully nonlinear, second-order, uniformly elliptic partial differential equations (PDEs for short):

$$F(x, Du, D^2u) = f \quad \text{in } \Omega, \tag{1}$$

where, $\Omega \subset \mathbf{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$ for simplicity, and $F : \Omega \times \mathbf{R}^n \times S^n \to \mathbf{R}$ and $f : \Omega \to \mathbf{R}$ are given functions. Here, S^n denotes the set of all symmetric $n \times n$ real matrices with the standard ordering.

It is well-known that **the Harnack inequality implies the Hölder continuity of solutions**. We note that this yields an equi-continuity of solutions since the Hölder exponent and the Hölder semi-norm depend only on the space-dimension, the uniform ellipticity constants and given data in (1).

This research is jointly done with N. S. Trudinger.

1.1 Hypotheses

In our mind, we consider the case when the coefficients of the second derivatives are merely measurable, and inhomogenious term belongs to only $L^n(\Omega)$. Moreover, we allow F to have the quadratic growth in the first derivatives.

However, F is supposed to be uniformly elliptic in the second derivatives.

Thus, our hypotheses are as follows:

Hypotheses

$$\begin{cases} (A1) \quad x \to F(x,p,X); \text{ measurable} & (p \in \mathbf{R}^n, X \in S^n), \\ (A2) \quad |F(x,p,O)| \le \gamma |p|^2 & (x \in \Omega, p \in \mathbf{R}^n), \\ (A3) \quad \mathcal{P}^-(X-Y) \le F(x,p,X) - F(x,p,Y) \le \mathcal{P}^+(X-Y) & \\ & (x \in \Omega, p \in \mathbf{R}^n, X, Y \in S^n), \\ (A4) \quad f \in L^n(\Omega), & \end{cases}$$

where, in $(A2)$, $\gamma > 0$ is a constant, and in $(A3)$, $\mathcal{P}^\pm : S^n \to \mathbf{R}$ are the so-called **Pucci operators** defined by

$$\mathcal{P}^+(X) = \max\{-\text{trace}(AX) \mid \lambda I \le A \le \Lambda I\},$$
$$\mathcal{P}^-(X) = \min\{-\text{trace}(AX) \mid \lambda I \le A \le \Lambda I\}.$$

In what follows, the above constants for uniform ellipticity $0 < \lambda \le \Lambda$ **are fixed.**

Under these hypotheses, we note that if u is a subsolution (resp., supersolution) of (1), then it is a subsolution (resp., supersolution) of

$$\mathcal{P}^-(D^2u) - \gamma|Du|^2 \le f \quad \left(\text{resp., } \mathcal{P}^+(D^2u) + \gamma|Du|^2 \ge f\right).$$

We will give the definition of sub- and supersolutions of (1) later.

It is immediate to see that the following properties on $\mathcal{P}^\pm$ hold true.

Proposition

(1) $\mathcal{P}^-(X) \le \mathcal{P}^+(X), \qquad \mathcal{P}^+(X) = -\mathcal{P}^-(-X), \qquad \mathcal{P}^\pm(\alpha X) = \alpha\mathcal{P}^\pm(X) \ (\alpha \ge 0)$

(2) $\mathcal{P}^-(X) + \mathcal{P}^-(Y) \le \mathcal{P}^-(X+Y) \le \mathcal{P}^+(X) + \mathcal{P}^-(Y) \le \mathcal{P}^+(X+Y) \le \mathcal{P}^+(X) + \mathcal{P}^+(Y)$

[Remark] In view of (1) and (2) in the above, it is easy to see that $\mathcal{P}^+$ is convex, and $\mathcal{P}^-$ is concave.

We shall give a typical example for which $(A1) - (A3)$ are satisfied.

《Example》

$$-\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + b(x)|Du|^2 = f(x) \tag{2}$$

Here, $A(\cdot) = (a_{ij}(\cdot))$, $b(\cdot)$ and $f(\cdot)$ satisfy the following:

$$\lambda|\xi|^2 \le \langle A(x)\xi, \xi\rangle \le \Lambda|\xi|^2 \ (\xi \in \mathbf{R}^n), \qquad \sup_{x\in\Omega}|b(x)| \le \gamma, \qquad f \in L^n(\Omega),$$

where $\langle\cdot,\cdot\rangle$ denotes the standard inner product in $\mathbf{R}^n$.

This kind of PDEs arises in the risk-sensitive stochastic control and certain PDEs derived from large deviation problems.

1.2 Known results

Let us mention known-results in case when the linear growth condition is supposed in place of $(A2)$;

$$|F(x,p,O)| \le \gamma|p| \quad (x, \in \Omega, p \in \mathbf{R}^n)$$

When F is merely measurable in x:

Krylov-Safonov [21] (1979) first obtained the Hölder continuity of solutions by a probabilistic approach. Trudinger [25] (1980) showed the same result as in [21] only by tools from PDEs. We note that in these results, solutions means "strong" solutions; they belong to $W^{2,n}_{loc}(\Omega)$ and satisfy the PDEs alomost everywhere sense.

Recently, Caffarelli [3] (1989) showed the Hölder continuity of the "standard" viscosity solutions when f is continuous but the estimate depends only on $\|f\|_{L^n(\Omega)}$. The reason why f is supposed to be continuous there is that Alexandroff-Bakelman-Pucci (ABP for short) maximum principle holds for the standard viscosity solutions only when $f \in C(\Omega)$. However, utilizing an approximation technique, Caffarelli-Crandall-Kocan-Święch [4] (1996) proved the ABP maximum principle when $f \in L^n(\Omega)$ for slightly restricted viscosity solutions.

In this article, we adapt the notion in [4], L^p-viscosity solutions, but, under the assumption $f \in C(\Omega)$, it is easy to check that our results below are still valid for the standard viscosity solutions.

For higher regularity of solutions, Caffarelli [3] obtained that soultions belong to $W^{2,n}_{loc}(\Omega)$ when "the oscillation of coefficients for the second derivatives are small in L^n-sense". However, in general, we cannot expect that solutions are in $W^{2,n}_{loc}(\Omega)$. Because, if we could get the higher regularity, then the solution would be the unique strong solution, which contradicts the fact that there exists a counter-example for uniqueness of viscosity solutions by Nadirashvili [22](1997). We also refer to Safonov [23](1999), which gives an alternative proof of [22] by a PDE approach.

When F is continuous in x:

Here, we only mention $C^{1,\alpha}$ ($\alpha \in (0,1)$) estimates for viscosity solutions by Trudinger [26] and [27].

1.3 Two ways to derive Harnack inequality

We recall the meaning that the **Harnack inequality holds**; For any $\Omega' \Subset \Omega$, there exists a constant $C = C(\mathrm{dist}(\Omega', \partial\Omega) > 0$ such that for any nonnegative solutions of (1), it follows that

$$\max_{\overline{\Omega'}} \leq C \left(\min_{\overline{\Omega'}} u + \mathrm{diam}(\Omega') \|f\|_{L^n(\Omega)} \right)$$

【Remark】 By the standard scaling argument and translation, we only have to show the above inequality when Ω' is a unit cube or a ball.

We shall use the following symbols:

$B_r := \{y \in \mathbf{R}^n \mid |y| \leq r\}$, $B_r(x) := B_r + x$, $Q_r := \{y \in \mathbf{R}^n \mid |y_k| \leq r/2\}$, $Q_r(x) := Q_r + x$

【Remark】 We notice the following inclusions hold.

$$Q_1 \subset B_{\sqrt{n}/2} \subset Q_{\sqrt{n}}.$$

《 difference of proofs bewteen Trudinger's and Caffarelli's》

Let us formally explain the difference of proofs between Trudinger's and Caffarelli's.

Trudinger's proof: We first derive the weak Harnack inequality for nonnegative **supersolutions** of (1). That is to find $\kappa > 0$ (possibly smaller than 1) and $C > 0$ such that

$$\|u\|_{L^\kappa(Q_1)} \leq C \left(\min_{Q_1} u + \|f\|_{L^n(Q_R)} \right)$$

for some $R > 1$ which only depends on n.

We remark that we obtain this estimate on cubes in place of balls since we essentially use Calderón-Zygmund's cube-decomposition lemma.

Next, we show the local maximum principle for nonnegative **subsolutions**; That is to find $C > 0$ such that with the above $\kappa > 0$ in the weak Harnack inequality for some $R > 1$,

$$\max_{B_1} u \leq C \left(\|u\|_{L^\kappa(B_R)} + \|f\|_{L^n(B_R)} \right)$$

Combining these, it is easy to show the Harnack inequality.

Caffarelli's proof: We use the (essentially) same argument as that of Trudinger to get the weak Harnack inequality for nonnegative **supersolutions**.

Next, for nonnegative **solutions**, we get a contradiction if we suppose that the Harnack inequality fails. To this end, we adapt a **blow-up arugument**. We note that we need properties of subsolutions and supersolutions.

2 Main results

Our aim is to show that any solutions of (1), for which assumptions $(A1)-(A3)$ are fulfilled, have the same equi-Hölder continuity. However, without further hypothesis, we cannot expect to prove such a result.

Let us present an example to show that we need further hypothesis.

《Example》 Let $n=1$ and $\Omega=(0,1)$. Set $u(x)=Ax$ $(A\geq 0)$. Notice that $-\triangle u=0$ in $(0,1)$. By setting $v(x)=e^{u(x)}$, it follows that

$$-\triangle v+e^{-Ax}|Dv|^2=0 \quad \text{in } (0,1)$$

Since $u\geq 0$ in $(0,1)$, the "coefficient" in front of $|Dv|^2$ is bounded for any $A\geq 0$. However, for any fixed small $\varepsilon\in(0,1/2)$, it is impposible to find $C=C(\varepsilon)>0$ independent of $A>0$ such that

$$\max_{x\in[\varepsilon,1-\varepsilon]} v(x)\leq C\min_{x\in[\varepsilon,1-\varepsilon]} v(x).$$

We notice that v is not "equi"-Hölder continuous when $A\to\infty$.

As will be seen, since our estimate depends only on λ,Λ,n and γ in the hytotheses, this example explains why we need the further hypothesis on L^∞-bound for solutions.

Now, we shall recall the definitions of viscosity solutions.

Throughout this article, we suppose that

$$(A5) \qquad p>\frac{n}{2}$$

Under $(A5)$, it is well-known that any function in $W^{2,p}_{loc}(\Omega)$ has second-order derivatives almost all $x\in\Omega$.

Definition

(1) We call $u\in C(\Omega)$ an L^p-viscosity subsolution (**subsolution** for short) of (1) if for any $\phi\in W^{2,p}_{loc}(\Omega)$, it follows that

$$\lim_{\varepsilon\to 0}\operatorname{ess}\inf_{y\in B_\varepsilon(x)}\{F(y,D\phi(y),D^2\phi(y))-f(y)\}\leq 0$$

provided $u - \phi$ attains its maximum at $x \in \Omega$.
(2) We call $u \in C(\Omega)$ an L^p-viscosity supersolution (**supersolution** for short) of (1) if for any $\phi \in W^{2,p}_{loc}(\Omega)$, it follows that

$$\lim_{\varepsilon\to 0} \operatorname{ess} \sup_{y\in B_\varepsilon(x)} \{F(y, D\phi(y), D^2\phi(y)) - f(y)\} \geq 0$$

provided $u - \phi$ attains its minimum at $x \in \Omega$.
(3) We call $u \in C(\Omega)$ an L^p-viscosity solution (**solution** for short) of (1) if it is an L^p-viscosity sub- and supersolution of (1).

【Remark】 We call u a C-viscosity (sub-, super-) solution if the above properties hold by replacing $W^{2,p}_{loc}(\Omega)$ by $C^2(\Omega)$. Since L^p-viscosity solutions are more restrictive than C-viscosity solutions, L^p-viscosity solutions are, indeed, C-viscosity solutions.
We remark that the opposite inclusion is true when F and f are continuous. See [4] for this fact.

We recall the notion of strong solutions here:

Definition

We call $u \in C(\Omega)$ a strong subsolution (resp., supersolution) of (1) if $Du(x)$ and $D^2u(x)$ exist for almost all $x \in \Omega$ and

$$F(x, Du(x), D^2u(x)) - f() \leq 0 \quad (\text{resp., } \geq 0) \quad a.e. \text{ in } \Omega.$$

We also call $u \in C(\Omega)$ a strong solution of (1) if it is a strong sub- and supersolution of (1).

In what follows, we mainly discuss about L^n-viscosity solutions.

Our main result is as follows:

Theorem

For any $N > 0$ and a subdomain $\Omega' \subset\subset \Omega$, if a solution u satisfies that $|u| \leq N$ in Ω, then there is $C > 0$ such that

$$\max_{B_r(x)} u \leq C\left(\min_{B_r(x)} u + r\|f\|_{L^n(\Omega)}\right) \quad (\text{for } x \in \Omega' \text{ and small } r > 0)$$

【Remark】 This result does not affect the counter-example.

We shall give a sufficient condition for (2) in the above example under zero-Dirichlet condition on $\partial\Omega$ for which the L^∞-estimate is a priori obtained.

《Example》

$$0 \le b(x) \le \gamma \quad \text{and} \quad f \ge 0 \quad \text{in } \Omega.$$

In fact, in this case, since 0 is a classical subsolution of (2), in view of the comparison principle between a strong subsolution and an L^n-supersolution in [18], we obtain that the L^n-solution u of (2) is nonnegative.

To obtain the upper bound, we find a strong supersolution $w \in C(\overline{\Omega}) \cap W^{2,n}_{loc}(\Omega)$ of

$$\begin{cases} \mathcal{P}^-(D^2w) \ge f^+ & a.e. \text{ in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \\ 0 \le w \le C\|f\|_{L^n(\Omega)} & \text{in } \Omega. \end{cases}$$

See the existence of strong solutions below for the proof of the existence of w. Since w is a strong supersolution of (2), the comparison principle again yields that

$$u \le w \le C\|f\|_{L^n(\Omega)} \quad \text{in } \Omega.$$

We modify the proof of Trudinger's in [27]. In fact, if we directly apply Caffarelli's blow-up argument, we can only succeed to prove the assertion in the case when the growth-order in p-variables is strictly less than 2. See [18] for this approach.

《Idea of proof》

(1) Use two different **transformations** to subsolutions and supersolutions, respectively, to simplify the original PDEs. (2) Show the local maximum principle for transformed subsolutions and the weak Harnack inequality for transformed supersolutions.

2.1 Preliminaries

Two key tools are the ABP maximum principle and the existence of strong solutions.

To this end, we introduce the **upper contact set** $\Gamma_\Omega[u]$ for $u \in C(\Omega)$;

$$\Gamma_\Omega[u] = \{x \in \Omega \mid \exists p \in \mathbf{R}^n \text{ such that } u(y) \le u(x) + \langle p, y-x\rangle \text{ for } \forall y \in \Omega\}$$

【Remark】 Roughly speaking, it holds that "$D^2u \le 0$"on $\Gamma_\Omega[u]$.

<u>**ABP maximum principle** (Proposition 3.3 in [4])</u>

Assume $f \in L^n(\Omega)$. There exists $C = C(\lambda, \Lambda, n, \Omega) > 0$ such that if $u \in C(\Omega)$ is an L^n-subsolution (resp., L^n-supersolution) of

$$\mathcal{P}^+(D^2u) \leq f \quad \left(\text{resp., } \mathcal{P}^+(D^2u) \geq f\right),$$

then it follows that

$$\max_{\overline{\Omega}} u^+ \leq \max_{\partial\Omega} u^+ + C\mathrm{diam}(\Omega)\|f^+\|_{L^n(\Gamma_\Omega[u^+])}$$

$$\left(\text{resp., } \max_{\overline{\Omega}} u^- \leq \max_{\partial\Omega} u^- + C\mathrm{diam}(\Omega)\|f^-\|_{L^n(\Gamma_\Omega[u^-])}\right)$$

【Remark】 (i) Here, we have used the notations:

$$u^+ := \max\{u, 0\} \quad \text{and} \quad u^- := \max\{-u, 0\}$$

(ii) If Ω is a ball or a cube, then $C > 0$ does not depend on Ω.
(iii) We do not know if this assertion holds true for C-solutions unless f is continuous.
(iv) The idea of proof is first to approximate f by smooth functions (see the proposition below for an existence result when f is smooth), and then, to approximate "u" by the sup-convolution (resp., inf-convolution) and the standard mollifier to apply the ABP maximum principle for strong solutions.

<u>Existence of strong solutions</u> <u>(Lemma 3.1 in [4])</u>

There exists $C > 0$ such that for $f \in L^n(\Omega)$, there is an L^n-strong subsolution $u \in C(\overline{\Omega}) \cap W^{2,n}_{loc}(\Omega)$ of

$$\begin{cases} (1) \quad \mathcal{P}^+(D^2u) \leq f & a.e. \text{ in } \Omega, \\ (2) \quad u = 0 & \text{on } \partial\Omega, \\ (3) \quad \|u\|_{L^\infty(\Omega)} \leq C\mathrm{diam}(\Omega)\|f\|_{L^n(\Omega)} & \end{cases}$$

【Remark】 (i) Here, the constant $C > 0$ is the one for the ABP maximum principle. We may have the corresponding result for $P^-(D^2u) \geq f$.
(ii) The sketch of proof is as follows: Choose $f_k \in C^\infty(\overline{\Omega})$ such that $\|f - f_k\|_{L^n(\Omega)} \to 0$, as $k \to \infty$. Since $\mathcal{P}^+$ is convex and independent of x, in view of [12], we know the existence of classical solutions u_k of

$$\begin{cases} \mathcal{P}^+(D^2u_k) = f_k & \text{in } \Omega, \\ u_k = 0 & \text{on } \partial\Omega. \end{cases}$$

Follwoing the argument in [4], we can get a uniform estimate for $\|u_k\|_{W^{2,n}_{loc}(\Omega)}$ and the uniform convergence to some $u \in C(\overline{\Omega})$. We remark that the limit function u only satisfies (1) since we may only know $u_k \rightharpoonup u$ weakly in $W^{2,n}_{loc}(\Omega)$.

2.2 Local maximum principle

Setting

$$w(x) = e^{\frac{\gamma u(x)}{\lambda}} - 1,$$

we observe that w is a nonnegative subsolution of

$$\mathcal{P}^-(D^2 w) \le \underline{f} := \frac{e^{\frac{\gamma u}{\lambda}} \gamma f}{\lambda}.$$

Since we suppose that $0 \le u \le N$, we do not have to worry about the right hand side of the above.

Local maximum principle

Fix any $p > 0$ and $\Omega' \subset\subset \Omega$. There exists $C = C(\lambda, \Lambda, n, \mathrm{dist}(\Omega', \partial\Omega), p) > 0$ such that

$$\max_{Q_r(x)} w \le C \left(\|w\|_{L^p(Q_{2\sqrt{n}r}(x))} + r\|\underline{f}\|_{L^n(Q_{2\sqrt{n}r}(x))} \right)$$

for $x \in \Omega'$ and small $r > 0$.

For simplicity, we shall obtain the assertion when $x = 0$ and $r = 1$. Let us write B for $B^o_{\sqrt{n}}$ for simplicity.

We first note that it is sufficient to show the case when $\underline{f} = 0$. Indeed, letting $\psi \in C(\overline{B}) \cap W^{2,n}_{loc}(B)$ be the strong subsolution of

$$\begin{cases} \mathcal{P}^+(D^2\psi) \le -\underline{f} & a.e. \text{ in } B, \\ \psi = 0 & \text{on } \partial B, \\ \|\psi\|_{L^\infty(B)} \le C\|\underline{f}\|_{L^n(B)}, & \end{cases}$$

we need to show the assertion for $w + \psi$, which is an L^n-subsolution of

$$\mathcal{P}^-(D^2(w + \psi)) \le 0.$$

We notice here that even for $p \in (0, 1)$, we have the following inequality in place of the triangle inequality for $p \ge 1$:

$$\|f_1 + f_2\|_{L^p(\Omega)} \le 2^{\frac{1}{p}} \left(\|f_1\|_{L^p(\Omega)} + \|f_2\|_{L^p(\Omega)} \right) \quad \text{for } f_1,\ f_2 \in L^p(\Omega).$$

Next, we introduce the following "cut-off" function:

$$\eta(|x|) = (n - |x|^2)^{\frac{2n}{p}}.$$

It is not hard to verify that $W(x) := \eta(x)w(x)$ satisfies

$$\mathcal{P}^-(D^2W) \leq C\left(\eta^{-\frac{p}{2n}}|DW| + \eta^{-\frac{p}{n}}W\right).$$

Since an easy geometrical observation implies that

$$|DW(x)| \leq \frac{W(x)}{\sqrt{n}-|x|} \leq C\eta^{-\frac{p}{n}}(x)W(x) \quad \text{for } x \in \Gamma_B[W^+].$$

Thus, since $Q_1 \Subset B_{\sqrt{n}}$, the ABP maximum principle yields that

$$\max_{Q_1} w \leq C\max_{B} W^+ \leq C\|\eta^{-\frac{p}{n}}W^+\|_{L^n(B)} \leq \frac{1}{2}\max_{B_1} W^+ + C\|w\|_{L^p(B)}.$$

More precisely, we first regularize w by the sup-convolution w^ε of it. Then,we get the estimate in a smaller ball B_r, where $r = r(\varepsilon) \to \sqrt{n}$ as $\varepsilon \to 0$.

[Remark] To deduce PDEs to homogenious ones, we need to work with L^n-solutions instead of C-solutions. In fact, we only know that the above ψ belongs to $W^{2,n}_{loc}(B)$ but $C^2(B)$.

2.3 Weak Harnack inequality

We shall adapt Caffarell's argument in [2] to show the weak Harnack inequality for supersoultions while in [19] we adapt the argument in [25].

We fisrt use the following transformation for u:

$$v(x) = 1 - e^{-\frac{\gamma u(x)}{\lambda}}$$

It is easy to see that v is a supersolution of

$$\mathcal{P}^+(D^2v) \geq \overline{f} := \frac{e^{-\frac{\gamma u}{\lambda}}\gamma f}{\lambda}.$$

Weak Harnack inequality
Fix $\Omega' \Subset \Omega$. There exist $p > 0$ and $C = C(\lambda, \Lambda, n, \Omega') > 0$ such that

$$\|v\|_{L^p(Q_r(x))} \leq C\left(\min_{Q_r(x)} v + r\|\overline{f}\|_{L^n(Q_{2\sqrt{n}r}(x))}\right)$$

for $x \in \Omega'$ and small $r > 0$.

As before, we may suppose that $x = 0$ and $r = 1$.

Again, considering $v+\psi$ instead of ψ, where $\psi \in C(\overline{B}_1)\cap W^{2,n}_{loc}(B_1)$ is a supersolution of

$$\begin{cases} \mathcal{P}^-(D^2\psi) \geq \overline{f}^- & a.e. \text{ in } B_1^o, \\ \psi = 0 & \text{on } \partial B_1, \\ 0 \leq \psi \leq C\|\overline{f}\|_{L^n(B_1)} & \text{in } B_1, \end{cases}$$

we only need to consider the case when $\overline{f}=0$.

Moreover, by considering $v(x)/(\min_{Q_1} v+\varepsilon)$ $(\varepsilon>0)$ instead of v, it is sufficient to find $p>0$ such that

$$\|v\|_{L^q(Q_1)} \leq C.$$

To this end, we need the following decay estimate of the distribution of v;

$$|\{x \in Q_1 \mid v(x) > t\}| \leq Ct^{-\tau} \quad (t \geq 0)$$

Here, $C>0$ and $\tau>0$ are independent of v. Thus, it suffices to show the following assertion for any integer $k \geq 1$:

$$|\{x \in Q_1 \mid v(x) > M^k\}| \leq \mu^k,$$

where $M>1$ and $\mu \in (0,1)$ are independent of k.

For the case of $k=1$, the above estimate is a direct consequence of the ABP maximum principle.

To show any $k \geq 1$, we argue by contradiction: Suppose that the assertion for k holds but fails for $k+1$. To get a contradiction, we use the cube-decomposition lemma by Calderón-Zygmund. See [2] for it.

2.4 Concluding remarks

In [19], following Escauriaza in [10], we give an extension to the case when f belongs to a slightly larger space, $L^p(\Omega)$ for some $p \in (n/2,n)$.

In [19], we also mention the Hölder estimate near the boundary, which ensures the global Hölder estimate. In a future work, we will discuss on higher regularity for solutions of (1) utlizing this global Hölder estimate.

【Open questions】 There must be so many open questions (at least to me) in this direction. We only list some of them:

(1) Harnack inequality near the boundary when $f \in L^p(\Omega)$ for $p \in (n/2,n)$. (i.e. Fabes-Stroock type formula near $\partial\Omega$.)

(2) Relation between Caffarelli's class and the VMO space for higher regularity.

(3) Sufficient conditions to show the existence of solutions of (1) in compariosn to Nagumo

condition.
(4) More delicate sufficient conditions to derive $L^\infty(\Omega)$ estimate than that mentioned here. (proposed by Prof. H. Nagai)
(5) More than quadratic nonlinearity. (proposed by Prof. M. Otani)
etc.

Though some of papers listed below are not refered here, for the interested readers, we give a list of related papers on L^p-solutions for fully nonlinear PDEs.

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